Exercise 6

Find the Laplace transform of the following expressions:

$$e^x - \cos x$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx.$$

Use the definition to find the answer.

$$\begin{aligned}
\mathcal{L}\{e^x - \cos x\} &= \int_0^\infty e^{-sx}(e^x - \cos x)\, dx \\
&= \int_0^\infty (e^{-sx}e^x - e^{-sx}\cos x)\, dx \\
&= \int_0^\infty e^{(-s+1)x}\, dx - \int_0^\infty e^{-sx}\cos x\, dx
\end{aligned}$$

Use Euler's formula to write cosine in terms of exponential functions.

$$\begin{aligned}
&= \frac{1}{-s+1} e^{(-s+1)x}\bigg|_0^\infty - \int_0^\infty e^{-sx}\left(\frac{e^{ix}+e^{-ix}}{2}\right) dx \\
&= \frac{1}{s-1} - \frac{1}{2}\left[\int_0^\infty e^{(-s+i)x}\, dx + \int_0^\infty e^{-(s+i)x}\, dx\right] \\
&= \frac{1}{s-1} - \frac{1}{2}\left[\frac{1}{-s+i} e^{(-s+i)x}\bigg|_0^\infty + \frac{1}{-(s+i)} e^{-(s+i)x}\bigg|_0^\infty\right] \\
&= \frac{1}{s-1} - \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) \\
&= \frac{1}{s-1} - \frac{1}{2} \cdot \frac{s+i+s-i}{(s-i)(s+i)} \\
&= \frac{1}{s-1} - \frac{1}{\not 2} \cdot \frac{\not 2 s}{s^2 - i^2}
\end{aligned}$$

Therefore,

$$\mathcal{L}\{e^x - \cos x\} = \frac{1}{s-1} - \frac{s}{s^2+1}.$$